1. STRUCTURE FUNCTIONS

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This section has been abridged from the full version of the *Review*.

1.1. Deep inelastic scattering

High-energy lepton-nucleon scattering (deep inelastic scattering) plays a key role in determining the partonic structure of the proton. The process $\ell N \to \ell' X$ is illustrated in Fig. 1.1. The filled circle in this figure represents the internal structure of the proton which can be expressed in terms of structure functions.

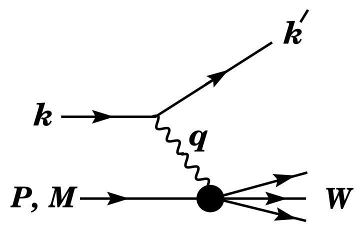

Figure 1.1: Kinematic quantities for the description of deep inelastic scattering. The quantities k and k' are the four-momenta of the incoming and outgoing leptons, P is the four-momentum of a nucleon with mass M, and W is the mass of the recoiling system X. The exchanged particle is a γ, $W^{\pm}$, or Z; it transfers four-momentum $q = k - k'$ to the nucleon.

Invariant quantities:

$\nu = \dfrac{q \cdot P}{M} = E - E'$ is the lepton's energy loss in the nucleon rest frame (in earlier literature sometimes $\nu = q \cdot P$). Here, E and E' are the initial and final lepton energies in the nucleon rest frame.

$Q^2 = -q^2 = 2(EE' - \overrightarrow{k} \cdot \overrightarrow{k}') - m_\ell^2 - m_{\ell'}^2$ where $m_\ell(m_{\ell'})$ is the initial (final) lepton mass. If $EE' \sin^2(\theta/2) \gg m_\ell^2, m_{\ell'}^2$, then

$\approx 4EE' \sin^2(\theta/2)$, where θ is the lepton's scattering angle with respect to the lepton beam direction.

$x = \dfrac{Q^2}{2M\nu}$ where, in the parton model, x is the fraction of the nucleon's momentum carried by the struck quark.

$y = \dfrac{q \cdot P}{k \cdot P} = \dfrac{\nu}{E}$ is the fraction of the lepton's energy lost in the nucleon rest frame.

$W^2 = (P + q)^2 = M^2 + 2M\nu - Q^2$ is the mass squared of the system X recoiling against the scattered lepton.

$s = (k + P)^2 = \dfrac{Q^2}{xy} + M^2 + m_\ell^2$ is the center-of-mass energy squared of the lepton-nucleon system.

The process in Fig. 1.1 is called deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS). In what follows, the masses of the initial and scattered leptons, m_ℓ and $m_{\ell'}$, are neglected.

1.1.1. *DIS cross sections*:

$$\frac{d^2\sigma}{dx\,dy} = x\left(s - M^2\right)\frac{d^2\sigma}{dx\,dQ^2} = \frac{2\pi\,M\nu}{E'}\frac{d^2\sigma}{d\Omega_{\rm Nrest}\,dE'}\ . \tag{1.1}$$

In lowest-order perturbation theory, the cross section for the scattering of polarized leptons on polarized nucleons can be expressed in terms of the products of leptonic and hadronic tensors associated with the coupling of the exchanged bosons at the upper and lower vertices in Fig. 1.1 (see Refs. 1–4)

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi y\alpha^2}{Q^4}\sum_j \eta_j\,L_j^{\mu\nu}\,W_{\mu\nu}^j\ . \tag{1.2}$$

For neutral-current processes, the summation is over $j = \gamma, Z$ and γZ representing photon and Z exchange and the interference between them, whereas for charged-current interactions there is only W exchange, $j = W$. (For transverse nucleon polarization, there is a dependence on the azimuthal angle of the scattered lepton.) $L_{\mu\nu}$ is the lepton tensor associated with the coupling of the exchange boson to the leptons. For incoming leptons of charge $e = \pm 1$ and helicity $\lambda = \pm 1$,

$$\begin{aligned}
L_{\mu\nu}^{\gamma} &= 2\left(k_\mu k'_\nu + k'_\mu k_\nu - k\cdot k' g_{\mu\nu} - i\lambda\varepsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta\right),\\
L_{\mu\nu}^{\gamma Z} &= (g_V^e + e\lambda g_A^e)\ L_{\mu\nu}^{\gamma}, \qquad L_{\mu\nu}^{Z} = (g_V^e + e\lambda g_A^e)^2\ L_{\mu\nu}^{\gamma},\\
L_{\mu\nu}^{W} &= (1 + e\lambda)^2\ L_{\mu\nu}^{\gamma},
\end{aligned} \tag{1.3}$$

where $g_V^e = -\frac{1}{2} + 2\sin^2\theta_W, \qquad g_A^e = -\frac{1}{2}$.

Although here the helicity formalism is adopted, an alternative approach is to express the tensors in Eq. (1.3) in terms of the polarization of the lepton.

The factors η_j in Eq. (1.2) denote the ratios of the corresponding propagators and couplings to the photon propagator and coupling squared

$$\begin{aligned}
\eta_\gamma &= 1 \quad ; \quad \eta_{\gamma Z} = \left(\frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha}\right)\left(\frac{Q^2}{Q^2 + M_Z^2}\right);\\
\eta_Z &= \eta_{\gamma Z}^2 \quad ; \quad \eta_W = \frac{1}{2}\left(\frac{G_F M_W^2}{4\pi\alpha}\,\frac{Q^2}{Q^2 + M_W^2}\right)^2 .
\end{aligned} \tag{1.4}$$

The hadronic tensor, which describes the interaction of the appropriate electroweak currents with the target nucleon, is given by

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z\,e^{iq\cdot z}\,\left\langle P,S\left|\left[J_\mu^\dagger(z), J_\nu(0)\right]\right|\,P,S\right\rangle, \tag{1.5}$$

where S denotes the nucleon-spin 4-vector, with $S^2 = -M^2$ and $S\cdot P = 0$.

1.2. Structure functions of the proton

The structure functions are defined in terms of the hadronic tensor (see Refs. 1–3)

$$\begin{aligned}
W_{\mu\nu} = &\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1\left(x, Q^2\right) + \frac{\hat{P}_\mu\hat{P}_\nu}{P\cdot q}\,F_2\left(x, Q^2\right)\\
&- i\varepsilon_{\mu\nu\alpha\beta}\,\frac{q^\alpha P^\beta}{2P\cdot q}\,F_3\left(x, Q^2\right)
\end{aligned}$$

$$+ \ i\varepsilon_{\mu\nu\alpha\beta}\,\frac{q^\alpha}{P\cdot q}\,\left[S^\beta g_1\left(x,Q^2\right) + \left(S^\beta \ - \ \frac{S\cdot q}{P\cdot q}\,P^\beta\right)\,g_2\left(x,Q^2\right)\right]$$

$$+ \ \frac{1}{P\cdot q}\,\left[\tfrac{1}{2}\left(\hat{P}_\mu\hat{S}_\nu \ + \ \hat{S}_\mu\hat{P}_\nu\right) - \frac{S\cdot q}{P\cdot q}\,\hat{P}_\mu\hat{P}_\nu\right]\,g_3\left(x,Q^2\right)$$

$$+ \ \frac{S\cdot q}{P\cdot q}\,\left[\frac{\hat{P}_\mu\hat{P}_\nu}{P\cdot q}\,g_4\left(x,Q^2\right) + \left(-g_{\mu\nu} \ + \ \frac{q_\mu q_\nu}{q^2}\right)\,g_5\left(x,Q^2\right)\right] \tag{1.6}$$

where

$$\hat{P}_\mu \ = \ P_\mu \ - \ \frac{P\cdot q}{q^2}\,q_\mu, \qquad \hat{S}_\mu \ = \ S_\mu \ - \ \frac{S\cdot q}{q^2}\,q_\mu \ . \tag{1.7}$$

The cross sections for neutral- and charged-current deep inelastic scattering on unpolarized nucleons can be written in terms of the structure functions in the generic form

$$\frac{d^2\sigma^i}{dxdy} \ = \ \frac{4\pi\alpha^2}{xyQ^2}\,\eta^i\left\{\left(1 \ - \ y \ - \ \frac{x^2y^2M^2}{Q^2}\right)\,F_2^i \right.$$
$$\left. + \, y^2xF_1^i \ \mp \ \left(y \ - \ \frac{y^2}{2}\right)\,xF_3^i\right\}, \tag{1.8}$$

where i = NC, CC corresponds to neutral-current ($eN \to eX$) or charged-current ($eN \to \nu X$ or $\nu N \to eX$) processes, respectively. For incoming neutrinos, $L^W_{\mu\nu}$ of Eq. (1.3) is still true, but with e,λ corresponding to the outgoing charged lepton. In the last term of Eq. (1.8), the $-$ sign is taken for an incoming e^+ or $\overline{\nu}$ and the $+$ sign for an incoming e^- or ν. The factor $\eta^{\rm NC} = 1$ for unpolarized $e^\pm$ beams, whereas

$$\eta^{\rm CC} \ = \ (1\pm\lambda)^2\,\eta_W \tag{1.9}$$

with $\pm$ for $\ell^\pm$; and where λ is the helicity of the incoming lepton and η_W is defined in Eq. (1.4); for incoming neutrinos $\eta^{\rm CC} = 4\eta_W$. The CC structure functions, which derive exclusively from W exchange, are

$$F_1^{\rm CC} = F_1^W, \ F_2^{\rm CC} = F_2^W, \ xF_3^{\rm CC} = xF_3^W . \tag{1.10}$$

The NC structure functions $F_2^\gamma, F_2^{\gamma Z}, F_2^Z$ are, for $e^\pm N \to e^\pm X$, given by Ref. 5,

$$F_2^{\rm NC} \ = \ F_2^\gamma \ - \ (g_V^e \pm \lambda g_A^e)\,\eta_{\gamma Z}F_2^{\gamma Z} \ + \ \left({g_V^e}^2 + {g_A^e}^2 \pm 2\lambda g_V^e g_A^e\right)\,\eta_Z F_2^Z \tag{1.11}$$

and similarly for $F_1^{\rm NC}$, whereas

$$xF_3^{\rm NC} = \ -\,(g_A^e \pm \lambda g_V^e)\,\eta_{\gamma Z}xF_3^{\gamma Z} + \left[2g_V^e g_A^e \pm \lambda\left({g_V^e}^2 + {g_A^e}^2\right)\right]\eta_Z xF_3^Z \ . \tag{1.12}$$

The polarized cross-section difference

$$\Delta\sigma \ = \ \sigma\left(\lambda_n = -1, \lambda_\ell\right) \ - \ \sigma\left(\lambda_n = 1, \lambda_\ell\right) \ , \tag{1.13}$$

where λ_ℓ, λ_n are the helicities (± 1) of the incoming lepton and nucleon, respectively, may be expressed in terms of the five structure functions $g_{1,\ldots 5}(x,Q^2)$ of Eq. (1.6). Thus,

$$\frac{d^2\Delta\sigma^i}{dxdy} \ = \ \frac{8\pi\alpha^2}{xyQ^2}\,\eta^i\left\{-\lambda_\ell y\left(2 - y - 2x^2y^2\frac{M^2}{Q^2}\right)\,xg_1^i + \lambda_\ell 4x^3y^2\,\frac{M^2}{Q^2}\,g_2^i\right.$$

$$
\begin{aligned}
&+\ 2x^2y\frac{M^2}{Q^2}\left(1-y-x^2y^2\frac{M^2}{Q^2}\right)\ g_3^i \\
&-\ \left(1+2x^2y\frac{M^2}{Q^2}\right)\left[\left(1-y-x^2y^2\frac{M^2}{Q^2}\right)\ g_4^i+xy^2g_5^i\right]\Bigg\} \qquad (1.14)
\end{aligned}
$$

with $i=$ NC or CC as before. In the $M^2/Q^2\to 0$ limit, Eq. (1.8) and Eq. (1.14) may be written in the form

$$
\begin{aligned}
\frac{d^2\sigma^i}{dxdy} &= \frac{2\pi\alpha^2}{xyQ^2}\,\eta^i\left[Y_+F_2^i\mp Y_-xF_3^i-y^2F_L^i\right], \\
\frac{d^2\Delta\sigma^i}{dxdy} &= \frac{4\pi\alpha^2}{xyQ^2}\,\eta^i\left[-Y_+g_4^i\mp Y_-2xg_1^i+y^2g_L^i\right], \qquad (1.16)
\end{aligned}
$$

with $i=$ NC or CC, where $Y_\pm = 1\pm(1-y)^2$ and

$$
F_L^i \;=\; F_2^i \;-\; 2xF_1^i\ , \qquad g_L^i \;=\; g_4^i \;-\; 2xg_5^i\ . \qquad (1.17)
$$

In the naive quark-parton model, the analogy with the Callan-Gross relations [6] $F_L^i=0$, are the Dicus relations [7] $g_L^i=0$. Therefore, there are only two independent polarized structure functions: g_1 (parity conserving) and g_5 (parity violating), in analogy with the unpolarized structure functions F_1 and F_3.

1.2.1. ***Structure functions in the quark-parton model***:

In the quark-parton model [8,9], contributions to the structure functions F^i and g^i can be expressed in terms of the quark distribution functions $q(x,Q^2)$ of the proton, where $q=u,\overline{u},d,\overline{d}$ *etc.*The quantity $q(x,Q^2)dx$ is the number of quarks (or antiquarks) of designated flavor that carry a momentum fraction between x and $x+dx$ of the proton's momentum in a frame in which the proton momentum is large.

For the neutral-current processes $ep\to eX$,

$$
\begin{aligned}
\left[F_2^\gamma,\ F_2^{\gamma Z},\ F_2^Z\right] &= x\sum_q\left[e_q^2,\ 2e_qg_V^q,\ g_V^{q\,2}+g_A^{q\,2}\right]\ (q+\overline{q})\ , \\
\left[F_3^\gamma,\ F_3^{\gamma Z},\ F_3^Z\right] &= \sum_q\left[0,\ 2e_qg_A^q,\ 2g_V^qg_A^q\right]\ (q-\overline{q})\ , \\
\left[g_1^\gamma,\ g_1^{\gamma Z},\ g_1^Z\right] &= \tfrac{1}{2}\sum_q\left[e_q^2,\ 2e_qg_V^q,\ g_V^{q\,2}+g_A^{q\,2}\right]\ (\Delta q+\Delta\overline{q})\ , \\
\left[g_5^\gamma,\ g_5^{\gamma Z},\ g_5^Z\right] &= \sum_q\left[0,\ e_qg_A^q,\ g_V^qg_A^q\right]\ (\Delta q-\Delta\overline{q})\ , \qquad (1.18)
\end{aligned}
$$

where $g_V^q=\pm\frac{1}{2}-2e_q\sin^2\theta_W$ and $g_A^q=\pm\frac{1}{2}$, with $\pm$ according to whether q is a $u-$ or $d-$type quark respectively. The quantity Δq is the difference $q\uparrow - q\downarrow$ of the distributions with the quark spin parallel and antiparallel to the proton spin.

For the charged-current processes $e^-p\to\nu X$ and $\overline{\nu}p\to e^+X$, the structure functions are:

$$
\begin{aligned}
F_2^{W^-} &= 2x\left(u+\overline{d}+\overline{s}+c\ldots\right)\ , \\
F_3^{W^-} &= 2\left(u-\overline{d}-\overline{s}+c\ldots\right)\ , \\
g_1^{W^-} &= \left(\Delta u+\Delta\overline{d}+\Delta\overline{s}+\Delta c\ldots\right)\ , \\
g_5^{W^-} &= \left(-\Delta u+\Delta\overline{d}+\Delta\overline{s}-\Delta c\ldots\right)\ , \qquad (1.19)
\end{aligned}
$$

where only the active flavors are to be kept and where CKM mixing has been neglected. For $e^+p\to\overline{\nu}X$ and $\nu p\to e^-X$, the structure functions F^{W^+}, g^{W^+} are obtained by the flavor interchanges $d \leftrightarrow u, s \leftrightarrow c$ in the expressions for F^{W^-}, g^{W^-}. The structure functions for scattering on a neutron are obtained from those of the proton by the interchange $u \leftrightarrow d$. For both the neutral- and charged-current processes, the quark-parton model predicts $2xF_1^i = F_2^i$ and $g_4^i = 2xg_5^i$.

Further discussion and all references may be found in the full *Review of Particle Physics*. The numbering of references and equations used here corresponds to that version.